One decomposition of hyperbolic unitary matrices

Vedran Šego

Faculty of Science, Department of Mathematics, University of Zagreb

7th Conference on Applied Mathematics and Scientific Computing
Trogir, Croatia, 13.6.2011.

Outline

Introduction

Notation

- $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$,
- $x, y, \dots \in \mathbb{F}^n$, $A, B, \dots \in \mathbb{F}^{m \times n}$,
- Euclidean scalar product: $\langle x, y \rangle := y^* x$,
- Hyperbolic scalar product induced by $J = \operatorname{diag}(\pm 1)$:
$$[x, y]_J := y^* J x.$$
When J is know from context: $[x, y]$,
- $\lambda(A)$ – set of eigenvalues of A,
- $\arg(z)$ argument (angle) of $z \in \mathbb{C}$,
- "norm" of x: $[x, x]$ – can be negative!

Hermitian matrices

- Euclidean adjoint (conjugate transpose) of A:
$$\langle Ax, y\rangle = \langle x, A^* y\rangle,$$
Hermitian matrix H:
$$H^* = H.$$
- Hyperbolic adjoint of A:
$$[Ax, y] = [x, A^{[*]}y], \text{ i.e. } A^{[*]} = JA^*J,$$
J-Hermitian matrix H:
$$H^{[*]} = H, \text{ i.e. } JH = HJ.$$
Matrix H is J-Hermitian if and only if JH is Hermitian.

Unitary matrices

- Euclidean unitary matrix U:

$$\langle Ux, Uy\rangle = \langle x, y\rangle, \text{ i.e. } \quad U^* U = \mathrm{I}, \; UU^* = \mathrm{I},$$

- J-unitary matrix:

$$[Ux, Uy] = [x, y],$$

i.e.

$$U^* JU = J, \quad U^{[*]} U = I, \quad UJU^* = J, \quad UU^{[*]} = I,$$

Note: Last two not true for more general J.

- Hyperexchange – "unitary-like" – matrix:

UP is J-unitary for some permutation P.

Importance: permutations are generally not J-unitary!

(Principal) square root

Defined for $A \in \mathbb{F}^{n \times n}, \lambda(A) \cap \mathbb{R}_0^- = \emptyset$.

$\sqrt{A} := X$, such that:

1. $X^2 = A$,
2. $\lambda(X) \subset \left\{ z \in \mathbb{C} : -\frac{\pi}{2} < \arg(z) < \frac{\pi}{2} \right\}$.

Such X is unique.

Properties

$$\sqrt{A} = \sum_{k=0}^{\infty} \binom{p}{k} (A - \mathrm{I})^k$$

$$\Rightarrow \sqrt{\cdot} \text{ is analytic}$$

$$\Rightarrow \sqrt{ABA} = A\sqrt{BA}.$$

(Principal) square root

Defined for $A \in \mathbb{F}^{n\times n}, \lambda(A) \cap \mathbb{R}_0^- = \emptyset$.

$\sqrt{A} := X$, such that:

1. $X^2 = A$,
2. $\lambda(X) \subset \left\{z \in \mathbb{C} : -\frac{\pi}{2} < \arg(z) < \frac{\pi}{2}\right\}$.

Such X is unique.

Properties

$$\sqrt{A} = \sum_{k=0}^{\infty} \binom{p}{k} (A - I)^k$$

$$\Rightarrow \sqrt{\cdot} \text{ is analytic}$$

$$\Rightarrow \sqrt{ABA} = A\sqrt{BA}.$$

Block-rotations

- Euclidean case:

$$\begin{bmatrix} \sqrt{I - XX^*} & X \\ -X^* & \sqrt{I - X^*X} \end{bmatrix},$$

- Hyperbolic case (Veselić):

$$\begin{bmatrix} \sqrt{I - XJ_2X^*J_1} & X \\ -J_2X^*J_1 & \sqrt{I - J_2X^*J_1X} \end{bmatrix}, \quad J = \operatorname{diag}(J_1, J_2).$$

Obviously, both are (J-)unitary.

SVD

- Euclidean case (traditional SVD):
  $A = U\Sigma V^*,\ U^*U = \mathrm{I},\ V^*V = \mathrm{I},\ \Sigma = \mathrm{diag}(\sigma_i), \sigma_i \geqslant 0,$
  Exists for any A.
- Hyperbolic case (2HSVD – two-sided hyperbolic SVD):
  $A = U\Sigma V^*,\ U^*JU = J,\ V^*JV = J,\ \Sigma = \mathrm{diag}(\sigma_i), \sigma_i J_{ii} \geqslant 0,$
  Exists if and only if $A^{[*]}A$ is $\mathbb{R}_0^+$-diagonalizable and A is nondegenerate ($\mathrm{rank}\, A = \mathrm{rank}\, A^*JA$).

Polar decomposition

- Euclidean case (traditional polar decomposition):
  $$A = UP, \quad U^* U = \mathrm{I},$$
  P is positive semidefinite Hermitian.
  Exists for any square A.
- Hyperbolic case (semidefinite J-polar decomposition by Bolshakov et al):
  $$A = UP, \quad U^* JU = J, \; P \text{ is } J\text{-nonnegative},$$
  i.e. JP is positive semidefinite.
  Exists under some complex conditions.

2HSVD and semmidefinite J-polar decomposition

J-polar decomposition + nondegeneracy $\Leftrightarrow$ 2HSVD.

Polar decomposition

- Euclidean case (traditional polar decomposition):
$$A = UP, \quad U^* U = \mathrm{I},$$
P is positive semidefinite Hermitian.
Exists for any square A.
- Hyperbolic case (semidefinite J-polar decomposition by Bolshakov et al):
$$A = UP, \quad U^* JU = J, \; P \text{ is } J\text{-nonnegative},$$
i.e. JP is positive semidefinite.
Exists under some complex conditions.

2HSVD and semmidefinite J-polar decomposition

J-polar decomposition + nondegeneracy $\Leftrightarrow$ 2HSVD.

QR decomposition

- Euclidean case:
$$A = QR, \quad Q^*Q = \mathrm{I}, \quad R \text{ upper triangular},$$
- Hyperbolic case (Singer):
$$A = P_1 QR P_2^*, \quad Q^* J' Q = J', \quad J' = P_1^* J P_1,$$
R is upper block-triangular with diagonal blocks of order 1 or 2.

Block operations

We shall decompose a J-unitary matrix U into a product of a block J-rotation and a block-diagonal J-unitary matrix:

$$U = R\Delta.$$

Generalization of a corresponding Euclidean result.

Example: Hyperbolic QR

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \text{one block rotation} \Rightarrow A'_{21} = 0$$

Block operations

We shall decompose a J-unitary matrix U into a product of a block J-rotation and a block-diagonal J-unitary matrix:

$$U = R\Delta.$$

Generalization of a corresponding Euclidean result.

Example: Hyperbolic QR

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \text{one block rotation} \ \Rightarrow \ A'_{21} = 0$$

Theory

1 Introduction

2 Theory
- Theorems
- Outlines of the proofs

3 Application

4 Conclusion

Euclidean case (Zakrajšek and Vidav)

For unitary

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

exist unitary

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} \Delta_{11} & \\ & \Delta_{22} \end{bmatrix}$$

such that $U = R\Delta$ and

$$R_{11} = \sqrt{I + R_{12}R_{21}}, \quad R_{21} = -R_{12}^*, \quad R_{22} = \sqrt{I + R_{21}R_{12}}.$$

Note: If U, R and Δ are of order 2, R is a (trigonometric) rotation.

Euclidean case (Zakrajšek and Vidav)

For unitary

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

exist unitary

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} \Delta_{11} & \\ & \Delta_{22} \end{bmatrix}$$

such that $U = R\Delta$ and

$$R_{11} = \sqrt{I + R_{12}R_{21}}, \quad R_{21} = -R_{12}^*, \quad R_{22} = \sqrt{I + R_{21}R_{12}}.$$

Note: If U, R and Δ are of order 2, R is a (trigonometric) rotation.

Hyperbolic case

For J-unitary

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad J = \mathrm{diag}(J_1, J_2),$$

if 2HSVDs of U_{11} and U_{22} exist, then there also exist J-unitary

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} \Delta_{11} & \\ & \Delta_{22} \end{bmatrix}$$

such that $U = R\Delta$ and

$$R_{11} = \sqrt{I + R_{12}R_{21}},\ R_{21} = -J_2 R_{12}^* J_1,\ R_{22} = \sqrt{I + R_{21}R_{12}}.$$

Note: If U, R and Δ are of order 2, R is a trigonometric (if $J_1 = J_2$) or hyperbolic rotation (if $J_1 = -J_2$).

Hyperbolic case

For J-unitary

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad J = \mathrm{diag}(J_1, J_2),$$

if 2HSVDs of U_{11} and U_{22} exist, then there also exist J-unitary

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} \Delta_{11} & \\ & \Delta_{22} \end{bmatrix}$$

such that $U = R\Delta$ and

$$R_{11} = \sqrt{I + R_{12}R_{21}}, \; R_{21} = -J_2 R_{12}^* J_1, \; R_{22} = \sqrt{I + R_{21}R_{12}}.$$

Note: If U, R and Δ are of order 2, R is a trigonometric (if $J_1 = J_2$) or hyperbolic rotation (if $J_1 = -J_2$).

Euclidean case

1. Polar decomposition: $U_{11} = R_{11}\Delta_1$, $U_{22} = R_{22}\Delta_2$.
2. U unitary $\Rightarrow R_{11} = \sqrt{I - R_{12}R_{12}^*}$, $R_{22} = \sqrt{I - R_{12}^*R_{12}}$.
   Also: $R_{11}(R_{12} + R_{21}^*) = 0$ and $(R_{12} + R_{21}^*)R_{22}^* = 0$.
3. If R_{11} or R_{22} nonsingular $\Rightarrow$ Q.E.D
4. Special case:

$$R_{11} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma \text{ positive real diagonal,}$$

   solved through some ugly formulas.
5. SVD: $R_{11} = V_1\Sigma_1 W_1^*$ and $R_{22} = V_2\Sigma_2 W_2^*$.
   Apply previous step $\Rightarrow$ extra unitary matrix on the left:

$$U = R'\Delta' = VR''(W^*\Delta').$$

   Using $\sqrt{ABA} = A\sqrt{BA}$ and some not-too-ugly formulas, V goes to the right hand side $\Rightarrow$ Q.E.D.

Euclidean case

1. Polar decomposition: $U_{11} = R_{11}\Delta_1$, $U_{22} = R_{22}\Delta_2$.
2. U unitary $\Rightarrow R_{11} = \sqrt{I - R_{12}R_{12}^*}$, $R_{22} = \sqrt{I - R_{12}^* R_{12}}$.
   Also: $R_{11}(R_{12} + R_{21}^*) = 0$ and $(R_{12} + R_{21}^*)R_{22}^* = 0$.
3. If R_{11} or R_{22} nonsingular $\Rightarrow$ Q.E.D
4. Special case:
$$R_{11} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma \text{ positive real diagonal},$$
   solved through some ugly formulas.
5. SVD: $R_{11} = V_1 \Sigma_1 W_1^*$ and $R_{22} = V_2 \Sigma_2 W_2^*$.
   Apply previous step $\Rightarrow$ extra unitary matrix on the left:
$$U = R'\Delta' = VR''(W^*\Delta').$$
   Using $\sqrt{ABA} = A\sqrt{BA}$ and some not-too-ugly formulas, V goes to the right hand side $\Rightarrow$ Q.E.D.

One decomp. of hyperbolic unitary matrices

V.Šego

Euclidean case

1. Polar decomposition: $U_{11} = R_{11}\Delta_1$, $U_{22} = R_{22}\Delta_2$.
2. U unitary $\Rightarrow R_{11} = \sqrt{I - R_{12}R_{12}^*}$, $R_{22} = \sqrt{I - R_{12}^*R_{12}}$.
   Also: $R_{11}(R_{12} + R_{21}^*) = 0$ and $(R_{12} + R_{21}^*)R_{22}^* = 0$.
3. If R_{11} or R_{22} nonsingular $\Rightarrow$ Q.E.D.
4. Special case:

$$R_{11} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma \text{ positive real diagonal,}$$

   solved through some ugly formulas.
5. SVD: $R_{11} = V_1\Sigma_1W_1^*$ and $R_{22} = V_2\Sigma_2W_2^*$.
   Apply previous step $\Rightarrow$ extra unitary matrix on the left:

$$U = R'\Delta' = VR''(W^*\Delta').$$

   Using $\sqrt{ABA} = A\sqrt{BA}$ and some not-too-ugly formulas, V goes to the right hand side $\Rightarrow$ Q.E.D.

Euclidean case

1. Polar decomposition: $U_{11} = R_{11}\Delta_1$, $U_{22} = R_{22}\Delta_2$.
2. U unitary $\Rightarrow R_{11} = \sqrt{I - R_{12}R_{12}^*}$, $R_{22} = \sqrt{I - R_{12}^* R_{12}}$.
   Also: $R_{11}(R_{12} + R_{21}^*) = 0$ and $(R_{12} + R_{21}^*)R_{22}^* = 0$.
3. If R_{11} or R_{22} nonsingular $\Rightarrow$ Q.E.D.
4. Special case:
   $$R_{11} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma \text{ positive real diagonal,}$$
   solved through some ugly formulas.
5. SVD: $R_{11} = V_1\Sigma_1 W_1^*$ and $R_{22} = V_2\Sigma_2 W_2^*$.
   Apply previous step $\Rightarrow$ extra unitary matrix on the left:
   $$U = R'\Delta' = VR''(W^*\Delta').$$
   Using $\sqrt{ABA} = A\sqrt{BA}$ and some not-too-ugly formulas, V goes to the right hand side $\Rightarrow$ Q.E.D.

Euclidean case

1. Polar decomposition: $U_{11} = R_{11}\Delta_1$, $U_{22} = R_{22}\Delta_2$.
2. U unitary $\Rightarrow R_{11} = \sqrt{I - R_{12}R_{12}^*}$, $R_{22} = \sqrt{I - R_{12}^*R_{12}}$.
   Also: $R_{11}(R_{12} + R_{21}^*) = 0$ and $(R_{12} + R_{21}^*)R_{22}^* = 0$.
3. If R_{11} or R_{22} nonsingular $\Rightarrow$ Q.E.D.
4. Special case:
   $$R_{11} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma \text{ positive real diagonal},$$
   solved through some ugly formulas.
5. SVD: $R_{11} = V_1\Sigma_1W_1^*$ and $R_{22} = V_2\Sigma_2W_2^*$
   Apply previous step $\Rightarrow$ extra unitary matrix on the left:
   $$U = R'\Delta' = \boldsymbol{V}R''(W^*\Delta').$$
   Using $\sqrt{ABA} = A\sqrt{BA}$ and some not-too-ugly formulas, V goes to the right hand side $\Rightarrow$ Q.E.D.

Hyperbolic case (polar decomposition)

1. Semidefinite J-polar decomposition:
$$U_{11} = R'_{11}\Delta'_1, \quad U_{22} = R'_{22}\Delta'_2.$$
2. U J-unitary $\Rightarrow$
Easy: $(R'_{11})^2 = I - R'_{12}J_2(R'_{21})^*J_1$.
But: $R'_{11} \neq \sqrt{I - R'_{12}J_2(R'_{21})^*J_1}$.
Instead:
$$R_{11} := R'_{11}(V_1J_1V_1^{[*]}) \text{ and } R_{22} := R'_{22}(V_2J_2V_2^{[*]})$$
V_1 and V_2 — from 2HSVD of R'_{11} and R'_{22}.
Then:
$R_{11}(R_{12}J_2 + J_1R_{21}^*) = 0$ and $(R_{12}J_2 + J_1R_{21}^*)R_{22}^* = 0$.
3. If R_{11} or R_{22} nonsingular $\Rightarrow$ Q.E.D.

Note: $V_1J_1V_1^{[*]} \neq J_1, I \quad (V_1J_1V_1^* = J_1,\ V_1V_1^{[*]} = I)$

Hyperbolic case (polar decomposition)

1. Semidefinite J-polar decomposition:
$$U_{11} = R'_{11}\Delta'_1, \quad U_{22} = R'_{22}\Delta'_2.$$
2. U J-unitary $\Rightarrow$
Easy: $(R'_{11})^2 = I - R'_{12}J_2(R'_{21})^* J_1$.
But: $R'_{11} \neq \sqrt{I - R'_{12}J_2(R'_{21})^* J_1}$.
Instead:
$$R_{11} := R'_{11}(V_1 J_1 V_1^{[*]}) \text{ and } R_{22} := R'_{22}(V_2 J_2 V_2^{[*]})$$
V_1 and V_2 — from 2HSVD of R'_{11} and R'_{22}.
Then:
$R_{11}(R_{12}J_2 + J_1 R^*_{21}) = 0$ and $(R_{12}J_2 + J_1 R^*_{21})R^*_{22} = 0$.
3. If R_{11} or R_{22} nonsingular $\Rightarrow$ Q.E.D.

Note: $V_1 J_1 V_1^{[*]} \neq J_1, I \quad (V_1 J_1 V_1^* = J_1, V_1 V_1^{[*]} = I)$

Hyperbolic case (polar decomposition)

1. Semidefinite J-polar decomposition:
$$U_{11} = R'_{11}\Delta'_1, \quad U_{22} = R'_{22}\Delta'_2.$$
2. U J-unitary $\Rightarrow$
Easy: $(R'_{11})^2 = I - R'_{12}J_2(R'_{21})^*J_1$.
But: $R'_{11} \neq \sqrt{I - R'_{12}J_2(R'_{21})^*J_1}$.
Instead:
$$R_{11} := R'_{11}(V_1 J_1 V_1^{[*]}) \text{ and } R_{22} := R'_{22}(V_2 J_2 V_2^{[*]})$$
V_1 and V_2 — from 2HSVD of R'_{11} and R'_{22}.
Then:
$$R_{11}(R_{12}J_2 + J_1 R^*_{21}) = 0 \text{ and } (R_{12}J_2 + J_1 R^*_{21})R^*_{22} = 0.$$
3. If R_{11} or R_{22} nonsingular $\Rightarrow$ Q.E.D.

Note: $V_1 J_1 V_1^{[*]} \neq J_1, I \quad (V_1 J_1 V_1^* = J_1,\ V_1 V_1^{[*]} = I)$

Hyperbolic case (polar decomposition)

1. Semidefinite J-polar decomposition:
$$U_{11} = R'_{11}\Delta'_1, \quad U_{22} = R'_{22}\Delta'_2.$$
2. U J-unitary $\Rightarrow$
Easy: $(R'_{11})^2 = I - R'_{12}J_2(R'_{21})^*J_1$.
But: $R'_{11} \neq \sqrt{I - R'_{12}J_2(R'_{21})^*J_1}$.
Instead:
$$R_{11} := R'_{11}(V_1 J_1 V_1^{[*]}) \text{ and } R_{22} := R'_{22}(V_2 J_2 V_2^{[*]})$$
V_1 and V_2 — from 2HSVD of R'_{11} and R'_{22}.
Then:
$$R_{11}(R_{12}J_2 + J_1 R_{21}^*) = 0 \text{ and } (R_{12}J_2 + J_1 R_{21}^*)R_{22}^* = 0.$$
3. If R_{11} or R_{22} nonsingular $\Rightarrow$ Q.E.D.

Note: $V_1 J_1 V_1^{[*]} \neq J_1, \mathrm{I} \quad (V_1 J_1 V_1^* = J_1, V_1 V_1^{[*]} = \mathrm{I})$

One decomp. of hyperbolic unitary matrices

V.Šego

Hyperbolic case (polar decomposition)

1. Semidefinite J-polar decomposition:
$$U_{11} = R'_{11}\Delta'_1, \quad U_{22} = R'_{22}\Delta'_2.$$
2. U J-unitary $\Rightarrow$
Easy: $(R'_{11})^2 = I - R'_{12}J_2(R'_{21})^*J_1$.
But: $R'_{11} \neq \sqrt{I - R'_{12}J_2(R'_{21})^*J_1}$.
Instead:
$$R_{11} := R'_{11}(V_1 J_1 V_1^{[*]}) \text{ and } R_{22} := R'_{22}(V_2 J_2 V_2^{[*]})$$
V_1 and V_2 — from 2HSVD of R'_{11} and R'_{22}.
Then:
$$R_{11}(R_{12}J_2 + J_1 R_{21}^*) = 0 \text{ and } (R_{12}J_2 + J_1 R_{21}^*)R_{22}^* = 0.$$
3. If R_{11} or R_{22} nonsingular $\Rightarrow$ Q.E.D.

Note: $V_1 J_1 V_1^{[*]} \neq J_1, \mathrm{I} \quad (V_1 J_1 V_1^* = J_1, V_1 V_1^{[*]} = \mathrm{I})$

Hyperbolic case (singular U_{11} and U_{22})

4. Special case:

$$R_{11} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma \text{ nonsingular real diagonal,}$$

solved through some ugly formulas.

Problem: 2HSVD does not groupe diagonal matrix (reminder: permutations are not always J-unitary).

4. Additional special case:

$$R_{11} = \operatorname{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_k),$$

Σ_i either zero or non-zero diagonal.
Solved using previous special case.

Hyperbolic case (singular U_{11} and U_{22})

4. Special case:

$$R_{11} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma \text{ nonsingular real diagonal,}$$

solved through some ugly formulas.

Problem: 2HSVD does not groupe diagonal matrix (reminder: permutations are not always J-unitary).

4. Additional special case:

$$R_{11} = \operatorname{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_k),$$

Σ_i either zero or non-zero diagonal.
Solved using previous special case.

Hyperbolic case (singular U_{11} and U_{22})

4. Special case:

$$R_{11} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma \text{ nonsingular real diagonal},$$

solved through some ugly formulas.

Problem: 2HSVD does not groupe diagonal matrix (reminder: permutations are not always J-unitary).

4. Additional special case:

$$R_{11} = \operatorname{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_k),$$

Σ_i either zero or non-zero diagonal.
Solved using previous special case.

Hyperbolic case (singular U_{11} and U_{22})

5. Like Euclidean case:
   2HSVD: $R_{11} = V_1\Sigma_1 W_1^{[*]}$ and $R_{22} = V_2\Sigma_2 W_2^{[*]}$
   Apply previous step $\Rightarrow$ extra unitary matrix on the left:
   $$U = R'\Delta' = \boldsymbol{V} R''(W^*\Delta').$$
   Using $\sqrt{ABA} = A\sqrt{BA}$ and some not-too-ugly formulas, V goes to the right hand side $\Rightarrow$ Q.E.D.

Application

Block-operations

- Limited use, due to strict conditions,
- Special cases when 2HSVD always exists:

$$J = \mathrm{diag}(\pm J_1, \pm J_2),$$

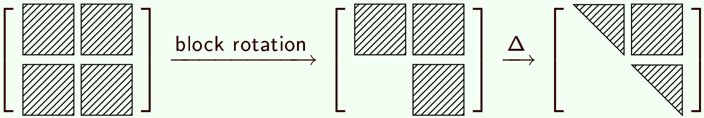
where J_k $(k = 1, 2)$ defines Euclidean $(J_k = \mathrm{I})$product.

- Idea:
  1. use R_{12} and/or R_{21} for block-elimination of hyperbolic parts,
  2. use Δ_1 and Δ_2 on Euclidean spaces.

Hyperbolic QR

- $P_1^* A P_2 = QR, \quad Q^* J' Q = J', \quad J' = P_1^* J P_1,$
- R is upper block-triangular with diagonal blocks of order 1 or 2.

The idea



Assume: there exists HQR such that

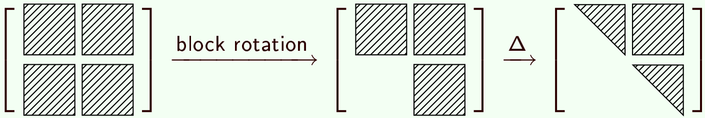
$J' = \mathrm{diag}(\mathrm{I}_k, -\mathrm{I}_{n-k})$ or, equivalently, $J' = \mathrm{diag}(-\mathrm{I}_k, \mathrm{I}_{n-k})$.

Note: Δ — from traditional (Euclidean) QR

Hyperbolic QR

- $P_1^* A P_2 = QR, \quad Q^* J' Q = J', \quad J' = P_1^* J P_1,$
- R is upper block-triangular with diagonal blocks of order 1 or 2.

The idea



Assume: there exists HQR such that

$J' = \mathrm{diag}(\mathrm{I}_k, -\mathrm{I}_{n-k})$ or, equivalently, $J' = \mathrm{diag}(-\mathrm{I}_k, \mathrm{I}_{n-k})$.

Note: Δ — from traditional (Euclidean) QR

Obtaining block rotation

We have:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad J' = \begin{bmatrix} \mathrm{I}_k & \\ & -I_{n-k} \end{bmatrix}.$$

Block-rotation:

$$B(X) := \begin{bmatrix} \sqrt{\mathrm{I}_k - XJ_2'X^*J_1'} & X \\ -J_2'X^*J_1' & \sqrt{\mathrm{I}_{n-k} - J_2'X^*J_1'X} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{\mathrm{I}_k - X(-\mathrm{I}_{n-k})X^*\mathrm{I}_k} & X \\ -(-\mathrm{I}_{n-k})X^*\mathrm{I}_k & \sqrt{\mathrm{I}_{n-k} - (-\mathrm{I}_{n-k})X^*\mathrm{I}_kX} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{\mathrm{I}_k + XX^*} & X \\ X^* & \sqrt{\mathrm{I}_k + X^*X} \end{bmatrix}$$

Note: J_1'- and J_2'-2HSVD ⇔ Traditional SVD!

One decomp. of hyperbolic unitary matrices

V.Šego

Obtaining block rotation

We need X such that

$$B(X)A = \begin{bmatrix} R'_{11} & R'_{12} \\ 0 & R'_{22} \end{bmatrix}.$$

In other words:

$$\begin{bmatrix} \sqrt{\mathrm{I}_k + XX^*} & X \\ X^* & \sqrt{\mathrm{I}_k + X^*X} \end{bmatrix} \cdot \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} R'_{11} & R'_{12} \\ 0 & R'_{22} \end{bmatrix},$$

i.e.

$$X^* A_{11} + \sqrt{\mathrm{I}_k + X^*X} A_{21} = 0.$$

Finding X

If A_{11} non-singular, let $A_{21}A_{11}^{-1} = V\Sigma W^*$ be (traditional) SVD of $A_{21}A_{11}^{-1}$.

Then $X = -W\Sigma\left(\sqrt{\mathrm{I}_k - \Sigma^2}\right)^{-1} V^*$, so

$$
\begin{aligned}
X^*A_{11} + \sqrt{\mathrm{I}_k + X^*X}A_{21} &= (X^* + \sqrt{\mathrm{I}_k + X^*X}A_{21}A_{11}^{-1})A_{11} \\
&= (X^* + \sqrt{\mathrm{I}_k + V\Sigma^2(I_k - \Sigma^2)^{-1}V^*}V\Sigma W^*)A_{11} \\
&= (X^* + V\Sigma\left(\sqrt{I_k - \Sigma^2}\right)^{-1} W^*)A_{11} \\
&= (X^* - X^*)A_{11} = 0.
\end{aligned}
$$

Note: we need $\left(\sqrt{\mathrm{I}_k - \Sigma^2}\right)^{-1}$

Finding X

Column "norms" (diagonal elements – must be nonnegative):

$$\begin{bmatrix} A_{11}^* & A_{21}^* \end{bmatrix} \begin{bmatrix} J_1' & \\ & J_2' \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} = A_{11}^* A_{11} - A_{21}^* A_{21}$$
$$= A_{11}^* (\mathrm{I}_k - (A_{21} A_{11}^{-1})^* (A_{21} A_{11}^{-1})) A_{11}$$
$$= A_{11}^* W (\mathrm{I}_k - \Sigma^2) W^* A_{11}$$

Left side positive definite $\Leftrightarrow \mathrm{I}_k - \Sigma^2 > 0$

$$\Rightarrow \exists \sqrt{\mathrm{I}_k - \Sigma^2}$$

$\longrightarrow$ Column permutations (P_2 in hyperbolic QR)

The algorithm

1. Find P_1 such that $J' =: P_1^* J P_1 = \mathrm{diag}(\mathrm{I}_k, \mathrm{I}_{n-k})$,
2. Find P_2 such that diagonal elements in
$$N := P_2^* A^* J' A P_2$$
have signs corresponding to J',
3. Check if the leading principal $k \times k$ submatrix $N[1:k, 1:k]$ is positive definite,
4. Find X (or, better, $B(X)$),
5. Apply block-rotation to obtain A' from A (such that the bottom left block is 0),
6. Apply traditional QR on diagonal blocks of A'.

Conclusion

To do

1. Expand the theory (add left and/or right permutations),
2. Find good algorithm for obtaining $B(X)$,
3. Find general block-HQR (for any J),
4. Apply the principle to other hyperbolic decompositions.

To do

1. Expand the theory (add left and/or right permutations),
2. Find good algorithm for obtaining $B(X)$,
3. Find general block-HQR (for any J),
4. Apply the principle to other hyperbolic decompositions.

To do

1. Expand the theory (add left and/or right permutations),
2. Find good algorithm for obtaining $B(X)$,
3. Find general block-HQR (for any J),
4. Apply the principle to other hyperbolic decompositions.

To do

1. Expand the theory (add left and/or right permutations),
2. Find good algorithm for obtaining $B(X)$,
3. Find general block-HQR (for any J),
4. Apply the principle to other hyperbolic decompositions.

Questions?

Vedran Šego, vsego@math.hr

Faculty of Science, Department of Mathematics, University of Zagreb